Chapter 5: Analysis of Time-Domain Circuits

This chapter begins the analysis of circuits containing elements with the ability to store energy: capacitors and inductors. We have already defined each of these reactive elements as an energy storage device with a differential IV relationship – we begin by using no more than these IV relationships and Kirchhoff’s laws to describe the time-domain solutions of DC switching circuits with capacitors and inductors.

Time-Domain Circuits using Differential Equations

The analysis of any DC circuit containing reactive elements will depend on solutions to differential equations, which in turn break down to the solution of a system’s homogeneous or natural response without any input and the heterogeneous or forced solution particular to a specific input. The form of the differential equation depends on the number of reactive components; the order of the differential equation (based in either a voltage or current) is equal to the number of capacitors and inductors. Typically, we are interested in some form of switching phenomenon; that is, at some fixed time (for simplicity translated to 0) a switch is either opened or closed changing the charging/discharging properties of the circuit elements. The result can be numerically calculated by solving the circuit’s associated differential equation. To set up these differential equations, we need to determine the set of equations (usually KVL loops or KCL sums) along with the appropriate initial and final conditions. Initial conditions are the values that a particular component(s) takes after a long time of being settled in the given state and just before switching at time zero. If this were the voltage in the capacitor before being hooked up to another circuit, we may represent it mathematically as \( V_C(t = 0^-) \). The final condition, often also called the steady state value, is the component’s value a long time after the switching phenomenon has taken place; the notation for such a capacitor voltage value would be \( V_C(t = \infty) \).

Capacitor and Inductor Time Properties

The most basic time-domain circuits with reactive elements consist of a current source charging a capacitor and a voltage source charging an inductor. The capacitor in Figure 5.1 is assumed to have an initial voltage \( V_C(t = 0) \) before the switch closes at time zero.

![Figure 1](image)

After the switch closes, positive charge flows from the current source to the capacitor effectively charging it. Using the time-domain IV relationship for the capacitor, we write the differential equation to obtain the capacitor voltage.

\[
I_{DC} = I_C(t) = C \cdot \frac{d}{dt}V_C(t)
\]

Resulting in:

\[
V_C(t) = \frac{I_{DC}}{C} \cdot t + V_C(t = 0)
\]

The second circuit shown in figure 5.2 is a voltage source in series with an inductor.
After the switch closes at time zero, the voltage source induces a constant voltage of $V_{DC}$ volts across the inductor, thereby causing a constantly increasing current.

\[
V_{DC} = V_L(t) = L \cdot \frac{d}{dt} I_L(t)
\]

\[
I_L = \frac{V_{DC}}{L} \cdot t
\]

Notice that neither one of these circuits is stable or fully realistic. As $t \to \infty$ both will store infinite amounts of energy; further, reopening the switch at any finite time leaves no path for current to flow and the reactive elements to discharge.
Example: Consider the charging/discharging of an initially uncharged $1 \mu F$ capacitor.

(a) Determine and plot the current for the capacitor if the voltage is 0 V for $t \leq 0$, $3t^2$ Volts for $0 < t < 2$, and 12 Volts for $t \geq 2$.

(b) Plot the instantaneous power of the capacitor.

(c) Plot the energy stored in the capacitor as a function of time.

(d) Verify that the instantaneous power is consistent with the changes in energy storage.

(a) Using the differential IV relationship to solve for the current $I_C(t)$.

\[ I_C(t) = (1 \mu F) \cdot \frac{d}{dt} V = \begin{cases} 0 & 0 \leq t \leq 0 \\ 3t^2 & 0 < t < 2 \\ 12 & t \geq 2 \end{cases} \]

(b,c) The instantaneous power is simply the product of voltage and current, while the energy stored in the capacitor can be most easily calculated as $E_{cap} = \frac{1}{2}CV^2$. Both waveforms are shown in Figure 5.4.

Exercise: Consider the charging/discharging of an initially unenergized ($I_L(t = 0^-) = 0A$) 1 mH inductor.
(a) Determine and plot the voltage across the inductor if the current is 0 A for \( t \leq 0 \), 2\( t \) Amps for \( 0 < t < 3 \), and 6 Amps for \( t \geq 3 \).

(b) Plot the instantaneous power of the inductor.

(c) Plot the energy stored in the inductor as a function of time.

(d) Verify that the instantaneous power is consistent with the changes in energy storage.

First-Order Circuits

To transition from the idealized energy storage properties shown above to more practical circuits, we must introduce series and parallel resistances (real voltage sources have some finite resistance, and the shunt resistance of any real component is large but not quite infinite). We will also see that these resistances affect the time-domain transient (initial or rapidly changing) response, but not the steady state values. We will start out with first-order circuits consisting of only one reactive element.

First-Order RC Circuits

The simplest first-order RC circuit is a voltage source in series with a single resistor and uncharged capacitor, as shown in Figure 5.5.

![Figure 5](image)

To analyze the first-order RC circuit, we need only write a single KVL loop equation. Since the relationship between the voltage and current in the capacitor is differential, this loop equation governing the system will inherently be a first-order differential equation, hence the title first-order circuit.

\[-V_x + V_R + V_C = 0\]

To solve a differential equation, we must have the entire equation in a single variable and its derivatives. We may substitute for \( V_R \) using Ohm’s law and the series current \( I_C(t) \) to obtain a term dependent on \( V_C \).

\[V_R(t) = R \cdot I_C(t) = R \cdot C \frac{d}{dt} V_C(t)\]

Replacing the original \( V_R \) by the term above yields the governing differential equation in one variable.

\[RC \frac{d}{dt} V_C(t) + V_C(t) = V_x\]

The final step before solving is to determine any initial or final conditions. Since the capacitor is uncharged, the initial voltage is zero; the final, or steady-state, voltage occurs when all time-derivatives are set to zero, giving \( V_C(t \rightarrow \infty) = V_x \). Another way to obtain this condition is to just look at the circuit: as long as the capacitor voltage is less than the source, positive current will flow clockwise through the resistor and the capacitor effectively charging the capacitor. This process will continue until the capacitor voltage is identical to the voltage source. Therefore,

\[V_C(t = 0^-) = 0 \quad V_C(t \rightarrow \infty) = V_x\]

To solve a first-order ordinary differential equation (ODE) we must consider a superposition of both the natural response and the forced response. The physical interpretation of the natural response is a characterization how the passive elements (series combination of resistor and capacitor) interact without any external input to the system, while the forced response is a linear
output of the passive system due to an external (or forcing) voltage/current source input. Solving each response individually results in the following.

Natural response:

\[ RC \frac{d}{dt} V_C(t) + V_C(t) = 0 \quad \Rightarrow \quad V_C(t) = \alpha e^{-\frac{t}{RC}} \]

Forced response:

\[ RC \frac{d}{dt} V_C(t) + V_C(t) = V_x \quad \Rightarrow \quad V_C(t) = \beta \]

Now, since the initial voltage on the capacitor is 0V, and the final voltage (we say \( t \rightarrow \infty \)) is \( V_x \), we may apply these two conditions to solve for the constants \( \alpha \) and \( \beta \).

\[ V_C(t) = V_x \cdot (1 - e^{-\frac{t}{RC}}) \quad \forall t \geq 0 \]

If, instead, we would like to solve for the voltage across the resistor, \( V_R \), we could simply subtract \( V_C \) from \( V_x \).

\[ V_R(t) = V_x - V_x \cdot (1 - e^{-\frac{t}{RC}}) \]
\[ = V_x \cdot e^{-\frac{t}{RC}} \quad \forall t \geq 0 \]

Or, alternately, we could calculate \( V_R \) from Ohms law and the differential IV relation of the capacitor.

\[ V_R(t) = R \cdot I_R(t) \]
\[ = R \cdot \left( C \frac{d}{dt} V_C(t) \right) \]
\[ = RC \cdot \frac{d}{dt} \left[ V_x \cdot (1 - e^{-\frac{t}{RC}}) \right] \]
\[ = RC \cdot V_x \cdot (-e^{-\frac{t}{RC}}) \cdot \left( -\frac{1}{RC} \right) \]
\[ = V_x \cdot e^{-\frac{t}{RC}} \quad \forall t \geq 0 \]

Corresponding plots of the voltage across the resistor and capacitor appear in Figure 5.6 (the voltage across the resistor is decaying exponentially as the capacitor charges).

**Figure 6**

First-Order RL Circuits

We will now repeat the differential equation analysis for the first-order RL circuit shown in Figure 5.7.
This time, we start by writing a single KCL equation at the top node, substituting the differential form of $I_L$ and using Ohm’s law to convert the resistor current to an inductor variable. We obtain another first-order ODE.

$$I_L = I_R + I_L = \frac{V_R}{R} + I_L = \frac{V_L}{R} + I_L = \frac{L}{R} \frac{d}{dt} I_L + I_L$$

Solving for the initial and final conditions, we see that the inductor current is initially zero (the switch is open) and that the final value is all of $I_x$. Remember that an inductor is basically a coil of wire, so at steady-state has zero voltage across it (corresponding to any constant value for the current through it).

$$I_L(t = 0^-) = 0 \text{A} \quad I_L(t \to \infty) = I_x$$

Solving the differential equation,

Natural response: \[ L \frac{d}{dt} I_L(t) + I_L(t) = 0 \] \[ \Rightarrow \quad I_L(t) = \alpha e^{-\frac{t}{L/R}} \]

Forced response: \[ L \frac{d}{dt} I_L(t) + I_L(t) = I_x \] \[ \Rightarrow \quad I_L(t) = \beta \]

and applying the initial and final conditions, we obtain our time-domain relations.

$$I_L(t) = I_x \cdot (1 - e^{-\frac{t}{L/R}}) \quad \forall t \geq 0$$

$$V_L(t) = I_x \cdot R \cdot e^{-\frac{t}{L/R}} \quad \forall t \geq 0$$

The waveforms for $I_L$ and $V_L$ look similar to the responses of the RC circuits and are shown in Figure 5.8.

An important fact to notice from our analyses of the first-order RC and RL circuits: a capacitor looks like a steady-state open circuit, while an inductor may be modeled as a steady-state short circuit. To prove this statement, we set the time derivatives to zero for each IV relationship obtaining specific conditions on the corresponding voltage and current. At steady-state,
\[ I_C = C \cdot \frac{d}{dt} V_C = C \cdot 0 = 0 \text{ Amps} \quad V_L = L \cdot \frac{d}{dt} I_L = L \cdot 0 = 0 \text{ Volts} \]

To guarantee zero current in the capacitor at steady-state, we must model it as an open circuit (allowing any constant steady-state voltage) and similarly guaranteeing zero voltage across the inductor by modeling it as a short circuit (allowing any constant steady-state current).

**Exercise:** Solve for the current through the 15Ω resistor in Figure 5.9 if the switch is closed at time zero. Assume that the capacitor is initially uncharged.

An **RC Switching Circuit**

Another example considers a more challenging first-order RC circuit, where the switch shown in Figure 5.10 opens and closes periodically (this is equivalent to a square-wave input voltage \( V_x \)).

Let us assume that there is no initial voltage on the capacitor, so \( V_C(t = 0) = 0 \text{ V} \). If we allow the circuit to come to rest (or steady state), we obtain the following final conditions.

**Switch closed:**
\[ V_C(t \to \infty) = \frac{R_2}{R_1 + R_2} \cdot V_x \]

**Switch open:**
\[ V_C(t \to \infty) = 0 \]

If we open the switch before the capacitor is fully charged, then the calculation will be the same as an initially charged capacitor discharging into \( R_2 \). We will use the differential substitutions:

\[ I_{R_2}(t) = \frac{V_C(t)}{R_2} \quad \text{and} \quad I_{R_1}(t) = I_{R_2}(t) + I_C(t) = \frac{V_C(t)}{R_2} + C \cdot \frac{d}{dt} V_C(t) \]

to obtain the two governing equations (depending on whether the switch is open or closed).

**Switch closed:**
\[ R_1 \cdot \left[ \frac{V_C(t)}{R_2} + C \cdot \frac{d}{dt} V_C(t) \right] + V_C(t) = V_x \]

**Switch open:**
\[ V_C(t) - R_2 \cdot C \cdot \frac{d}{dt} V_C(t) = 0 \]
The corresponding solutions are:

Switch closed: \[ V_C(t) = \frac{R_2}{R_1 + R_2} \cdot V_x \cdot (1 - e^{-\frac{t}{\tau RC}}) \]

Switch open: \[ V_C(t) = V_{init} \cdot e^{-\frac{t}{\tau RC}} \]

where \( V_{init} \) is the voltage on the capacitor when the switch is opened.

You should verify that the capacitor is charging when the switch is closed and discharging when the switch is open. Further, you should notice that the time constant (the time constant, \( \tau \), is the rate at which the exponential function decays—for RC circuits \( \tau = RC \), and for RL circuits, \( \tau = L/R \)) is different, and thus the capacitor charges and discharges at different rates. The plot shown in Figure 5.11 corresponds to a square wave of frequency 2\( \pi \).

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**Extension of Time-Domain Circuits**

Clearly, the analyses of time-domain circuits presented in this chapter have been exceedingly tedious, even for small circuits. A simple electrical design may contain circuits with 10 reactive components, requiring a 10\( \text{th} \) order differential equation be solved. Also, we are more likely to have input signals that are sinusoidal in nature, a much more daunting computational task. Computers are good at performing such calculations, but engineers would quickly become too bogged down to make the analysis worthwhile.

Consider the simple first-order RC circuit again, but this time allow the input voltage to have a value \( A \cos(\omega t) \), resulting in a new differential equation.

\[ RC \frac{d}{dt} V_C(t) + V_C(t) = V_x = A \cdot \cos(\omega t) \]

Repeating our solutions of the forced response, we obtain the steady-state capacitor voltage.

\[ V_{C,ss}(t) = \frac{A}{1 + (\omega RC)^2} \cdot \cos(\omega t) + \frac{A \cdot \omega RC}{1 + (\omega RC)^2} \cdot \sin(\omega t) \]

Rather than trudge your way through the math to duplicate this result for the other circuits, make note of the fact that a sinusoidal input (or forcing function) yields a frequency-dependent sinusoidal output. We are not including any natural response from the system, but since that response is a decaying exponential, it will not affect the overall response after a few seconds. Also notice that the amplitudes of the output sinusoid(s) depend on the frequency of the input sinusoid—this will allow us to move on to more powerful analysis techniques.

**Summary**

In place of larger and more complicated time-domain analyses, we will transfer our analysis of time-varying circuits to a steady-state analysis in the frequency domain (Chapter 7), where all our equations reduce from differential to algebraic. The limitation of the frequency-domain analysis is that the transient information is lost in the conversion—we only obtain a solution based upon the steady-state response. Before starting our analysis of these reactive circuits in the frequency-domain, we will explain complex numbers and complex-variable theory in a little more detail (Chapter 6).
Problem 5.1: A 10\mu F capacitor used as a DC blocking capacitor has a voltage applied to it according to:

\[
\begin{align*}
V_C(t) &= 1 + 2t \text{ V}\quad 0 \text{ms} \leq t < 3 \text{ms} \\
V_C(t) &= 19 - 4t \text{ V}\quad 3 \text{ms} \leq t < 4 \text{ms} \\
V_C(t) &= 3 \text{ V}\quad 4 \text{ms} \leq t
\end{align*}
\]

(a) Determine the initial conditions for the voltage across the capacitor; that is, solve for \(V_C(t = 0^-)\).
(b) Plot the capacitor’s voltage, current, instantaneous power, and energy storage as a function of time – show graphically that energy may be derived from an equation or by integrating the power.

Problem 5.2: A parallel-plate capacitor having Silicon-Dioxide between the plates as a dielectric \((\epsilon_r, \varepsilon_{\text{SiO}_2} \approx 11.7)\) and a separation distance of 1 mm stores an average 4 mJ of energy when \(V_C(t) = 1 + 2\cos(100t) \text{ V}\). Determine the equivalent value of the capacitance and the plate dimensions both with and without the dielectric.

Problem 5.3: Consider a 3.3 mH inductor used in a audio filter initially stores 10 mJ in an electric field and has a time-domain voltage across of:

\[
\begin{align*}
V_L(t) &= 15 + 50\cos(100t) \text{ mV}\quad 0 \text{ms} \leq t < 10 \text{ms} \\
V_L(t) &= 42 \text{ mV}\quad 10 \text{ms} \leq t < 20 \text{ms} \\
V_L(t) &= 0 \text{ V}\quad 20 \text{ms} \leq t
\end{align*}
\]

(a) Determine the initial conditions for the current through and the voltage across the inductor.
(b) Plot the inductor’s voltage, current, instantaneous power, and energy storage as a function of time – show graphically that energy may be derived from an equation or by integrating the power.

Problem 5.4: The voltage of the capacitor shown in Figure 5.14 has an instantaneous value of

\[
\begin{align*}
V_C(t) &= 4t \text{ V}\quad 0 \leq t < 2 \text{s} \\
V_C(t) &= 8 \text{ V}\quad 2 \leq t < 4 \text{s} \\
V_C(t) &= 24 - 4t \text{ V}\quad 4 \leq t < 6 \text{s} \\
V_C(t) &= 0 \text{ V}\quad 6 \leq t
\end{align*}
\]

(a) Determine the initial conditions for the current through and the voltage across the inductor.
(b) Plot the inductor’s voltage, current, instantaneous power, and energy storage as a function of time – show graphically that energy may be derived from an equation or by integrating the power.
Problem 5.5: The inductor shown in Figure 5.15 has an instantaneous current of
\[ I_L(t) = \begin{cases} 
-2 t & 0 \leq t < 1 \\
-2 & 1 \leq t < 2 \\
-10 + 4 t & 2 \leq t < 4 
\end{cases} \text{ mA} \]

Figure 15

Problem 5.6: How would you mathematically model an instantaneous transition in the voltage across a capacitor or the current through an inductor? What about the corresponding current and voltage, respectively?

Problem 5.7: Derive and solve the first-order differential equation governing the circuit shown in Figure 5.16. Specifically, solve for the capacitor voltage as a function of time both before and after the switched has opened or closed.

Problem 5.8: Derive and solve the first-order differential equation governing the circuit shown in Figure 5.17. Specifically, solve for the inductor current as a function of time both before and after the switched has opened or closed.
Problem 5.9: Without solving the problems in their entirety, what are the time constants of the circuits shown in Figures 5.13 and 5.14? Give answers for the switch open and the switch closed (total of four time constants).

Problem 5.10: Derive and solve the first-order differential equation governing the circuit shown in Figure 5.18. Specifically, solve for the inductor voltage as a function of time both before and after the switch has opened or closed.

Problem 5.11: Derive and solve the first-order differential equation governing the circuit shown in Figure 5.19. Specifically, solve for the capacitor current as a function of time both before and after the switch has opened or closed.